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A Penalty Function for Enforcing Maximum Length Scale Criterion in Topology Optimization

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[Abstract] Techniques for imposing a minimum length scale, or minimum feature size, in continuum topology optimization have been proposed in literature, with the motivation being stabilization of the maximum stiffness problem and satisfying manufacturing constraints. Imposing an upper bound on feature sizes, however, has not been investigated. This paper proposes a scheme for restricting the maximum length scale in topology optimization. In short, the design domain is searched and structural features larger than the prescribed maximum length scale are penalized. The scheme is implemented in the context of minimum compliance design together with an existing minimum length scale methodology. The resulting optimization problem is continuous and solved using the Method of Moving Asymptotes (MMA). Beam design examples are considered and solutions are shown to be near 0/1 (void/solid) topologies that satisfy the minimum and maximum length scale criteria. The designer thus gains complete control over member sizes and therefore additional influence over cost and manufacturability. Further, restricting maximum length scale can potentially be used to introduce structural redundancy into the design, as loads previously carried by few large members are often redistributed over an interconnected system of smaller members.

I. Introduction

SEVERAL researchers have pursued techniques for achieving a prescribed minimum length scale, or minimum feature size, in continuum topology optimization[1-4]. The primary motivation for these works is stabilization of the minimum compliance (maximum stiffness) design problem, which is known to generally lack solutions. Designs can often be improved without changing the total volume of material by reducing the size and increasing the number of holes in the topology. This leads to chattering designs where the number of microscopic holes becomes unbounded[5]. When solving these problems numerically, this characteristic is revealed in the form of mesh dependence, where solutions change with mesh refinement, and checkerboard patterns, regions that rapidly vary between solid and void phases.

The minimum compliance problem can be stabilized by restricting the design space. One such approach is to require structural features to attain a minimum length scale. Such a restriction prevents thinner members from developing when using smaller mesh sizes and forbids the formation of checkerboard patterns. Additionally, imposing a minimum length scale allows designers to consider manufacturing constraints and cost. More intricate designs featuring thinner members with a greater number of connections are generally more costly and difficult to construct.

This work investigates the additional restriction of imposing a maximum length scale on structural features, thereby giving the designer complete control over member sizes and further influence over manufacturability and cost. Prescribing a maximum length scale also potentially offers a means for implementing structural redundancy in

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elastic problems or maximum pore size in fluid device design. In general, large, dominating features that are greater than the maximum length scale are optimally redistributed as an interconnected system of smaller features.

The upper bound on feature size is enforced via a penalty expression in the objective function of the topology optimization problem. The design domain is searched and violations of maximum length scale are penalized via an exponential function. The technique is demonstrated on traditional minimum compliance design examples with minimum length scale controlled through the nodal design variable and projection function approach[4]. The proposed scheme is shown to yield nearly 0/1 (void/solid) solutions that satisfy the prescribed minimum and maximum length scale criteria.

The layout of this paper is as follows. Section 2 describes the continuum maximum length scale penalty function and discusses its implementation in the context of the minimum compliance problem. Section 3 presents solutions to design examples and Section 4 offers concluding remarks.

II. Maximum Length Scale Formulation

The proposed maximum length scale constraint requires the diameter d of all structural members to be less than $2r_{max}$, where the length scale r_{max} is the upper bound on the radius of structural members. The minimum radius of structural members will be denoted by r_{min}, and consequently $2\, r_{min} \le d < 2\, r_{max}$. As r_{min} and r_{max} are physical length scales they do not change with mesh refinement. The scheme proposed here enforces this constraint by passing a circular region Ω_{rx} of radius r_{max} over the entire design domain Ω, and checking that this region is never completely filled with solid material:

$$\int_{\Omega_{rx}} \rho(x)dx < \int_{\Omega_{rx}} dx \quad \textit{everywhere in the domain } \Omega \tag{1}$$

where $\rho(\mathbf{x})$ is the traditional material distribution function with $\rho(\mathbf{x})=1$ indicating solid material present at location $\mathbf{x}$ and $\rho(\mathbf{x})=0$ indicating the presence of a void.

In other words, the circular region Ω_{rx} must always contain some quantity of voids. For discretized problem formulations, the circular sub-domain can be passed over the entire domain by centering Ω_{rx} at the centroid of each element. Figure 1 illustrates this concept for an arbitrary element e, where the sub-domain of radius r_{max} is now denoted as Ω^e_{rx}. The material distribution function is defined as constant within each element and will be denoted as the traditional ρ^e, referred to as the element volume fraction, or relative density. We will be using the nodal design variable and projection function scheme developed by the authors for imposing a minimum length scale[4]. This technique has been shown to yield 0/1, minimum length scale compliant topologies for a variety of problems. In short, nodal design variables ρ_n are the design variable and are projected onto element space using regularized Heaviside functions to determine the element volume fractions ρ^e. Consequently, ρ^e are expressed as a function of ρ_n: $\rho^e(\rho_n)$. For details on this scheme, the reader is referred to Reference 4.

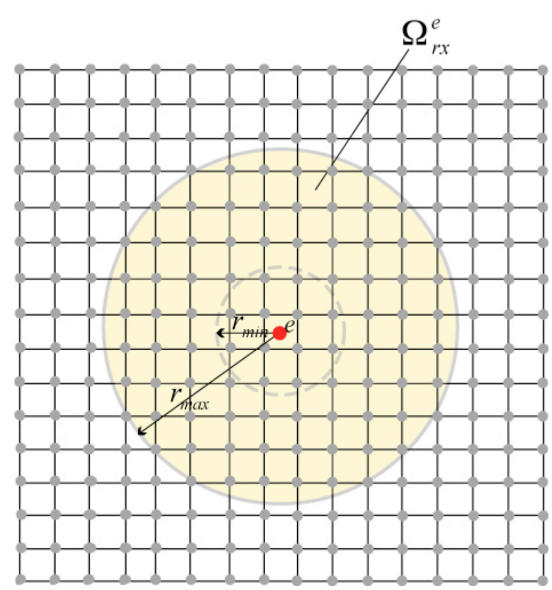

Figure 1. The domain Ω^e_{rx} is checked for maximum length scale violation.

The maximum length scale constraint should be constructed so that it is independent of mesh size. Further, it should not reward intermediate volume fractions when solving continuous topology optimization formulations where the binary constraint on the design variable ρ^e is relaxed. In such formulations, intermediate values of ρ^e are permitted but are considered undesirable and are thus penalized to drive the solution to a 0/1 topology. For example, the popular SIMP method[6] penalizes the stiffness of intermediate volume fractions, values between 0 and 1, by scaling element stiffness matrices by $(\rho^e)^p$, with $p > 1$. Intermediate volume fractions should be treated in a similar manner by the maximum length scale constraint so as not to counteract such penalties. It is difficult to respect these properties when employing a limit on the volume of solids contained in Ω^e_{rx}. Therefore, the constraint is formulated to require a minimum volume voids (V_v^{min}) to be located inside Ω^e_{rx}:

$$V_v^e \geq V_v^{min} \qquad \forall\, e \in \Omega \tag{2}$$

where V_v^e is the actual volume of voids located inside Ω^e_{rx}. To minimize mesh dependency, V_v^{min} is computed as a percentage of the total volume of Ω^e_{rx}:

$$V_v^{min} = c_v^{min} \sum_{i \in \Omega^e_{rx}} v^i \tag{3}$$

where v^i is the volume of element i and c_v^{min} is the user-defined percentage of voids required ($c_v^{min} > 0$). For example, the results shown later in this section were produced using c_v^{min} = 2%. For non-uniform meshes, V_v^{min} may not be the same for all elements and thus should be computed for each element.

The actual volume of voids contained in Ω^e_{rx} is computed via the following expression:

$$V_v^e(\boldsymbol{\rho}_\mathbf{n}) = \sum_{i \in \Omega^e_{rx}} v^i (1 - \rho^i(\boldsymbol{\rho}_\mathbf{n}) + \rho^e_{min})^\eta \tag{4}$$

where exponent η limits the degree to which elements with intermediate volume fractions contribute to the volume of voids computation ($\eta \geq 1$) and ρ^e_{min} is the minimum allowable element volume fraction. For large values of η, only elements with volume fractions that achieve the lower bound ($\rho^e = \rho^e_{min}$) are counted towards the total volume of voids. Elements with volume fractions above ρ^e_{min} will yield a value less than 1 inside the parenthesis of Eq. (4), which approaches 0 when raised to large η. Therefore, only true void elements will contribute to the computation. Note that when continuation methods are used to gradually increase the penalty on intermediate volume fractions (such as the SIMP method), η should initially be set equal to one as intermediate volume fractions are not yet penalized. As the optimization algorithm progresses, η is gradually raised, much like the exponent p used by the SIMP method.

A. A Maximum Length Scale Penalty Function

Requiring a minimum volume of voids in each elemental domain Ω^e_{rx} introduces a large number of nonlinear constraints into the optimization problem. To circumvent this drawback the maximum length scale requirement is imposed via penalization of the objective function, rather than through explicit constraints. The penalty function $S_{rx}(\boldsymbol{\rho}_\mathbf{n})$ is added to minimization objective functions and thus should take on the following properties:

$$\begin{aligned} S_{rx}(\boldsymbol{\rho}_\mathbf{n}) &= \sum_{e \in \Omega} S^e_{rx}(\boldsymbol{\rho}_\mathbf{n}) \\ S^e_{rx}(\boldsymbol{\rho}_\mathbf{n}) &= 0 \qquad \mathit{if} \quad c^e_v(\boldsymbol{\rho}_\mathbf{n}) \geq c_v^{min} \\ S^e_{rx}(\boldsymbol{\rho}_\mathbf{n}) &>> 0 \qquad \mathit{if} \quad c^e_v(\boldsymbol{\rho}_\mathbf{n}) < c_v^{min} \end{aligned} \tag{5}$$

where the void constraint has been normalized by expressing it in terms of void ratios, with $c^e_v(\boldsymbol{\rho}_\mathbf{n})$ representing the actual percentage of Ω^e_{rx} filled with voids:

$$c^e_v(\boldsymbol{\rho}_\mathbf{n}) = \frac{V^e_v(\boldsymbol{\rho}_\mathbf{n})}{\sum_{i \in \Omega^e_{rx}} v^i} \tag{6}$$

Note that S_{rx}^{e} is to be constructed so that it is never less than zero. If permitted to take on negative values, the penalty function would reward designs that use thinner members (far from r_{max}), or give the illusion of using thinner members by featuring intermediate volume fractions, potentially counteracting the SIMP penalization on stiffness. While this may be of interest in other works, we will limit this technique to simply bounding member sizes.

Eq. (5) is discrete and must be regularized before applying to continuous optimization problems. One possible regularization is characterized by the curves in Figure 2 and defined as

$$S_{rx}^{e}(\boldsymbol{\rho}_{\mathbf{n}}) = \left(\left(1 + c_v^{min} - \alpha_1\right)\left(1 - c_v^{e}(\boldsymbol{\rho}_{\mathbf{n}})\right)\right)^{\alpha_2} \quad (7)$$

where α_1 and α_2 control the location and slope of the curve, respectively, as illustrated by Figure 2. Note that the function is appropriately zero when Ω_{rx}^{e} contains only void elements ($c_v^{e} = 1$).

When the constraint is exactly satisfied ($c_v^{e}(\boldsymbol{\rho}_{\mathbf{n}}) = c_v^{min}$), the penalty function should achieve a magnitude close to zero for large α_2. It follows that the following condition must hold:

$$\alpha_1 \;>\; \frac{-\left(c_v^{min}\right)^2}{1 - c_v^{min}} \quad (8)$$

Like the penalty on intermediate volume fractions, a continuation method should be applied to the maximum length scale penalty so that penalization increases as the optimization algorithm progresses and approaches a 0-1 solution. This requires decreasing α_1 to move the high penalty areas of the curve closer to the bound c_v^{min} and raising α_2 to increase the magnitude of the penalization. Figure 3 demonstrates this technique as the curve is moved from (a) to (d).

Eq. (9) is just one possible regularization of Eq. (5). It is interesting to note that this regularization function was also recently used to impose a minimum permeability in the design of multifunctional material microstructures optimized for stiffness and permeability[7].

B. Solving the minimum compliance problem with minimum and maximum length scale criteria

The minimum compliance optimization problem is given in the minimum potential energy formulation with nodal design variables[4] and maximum length scale control as

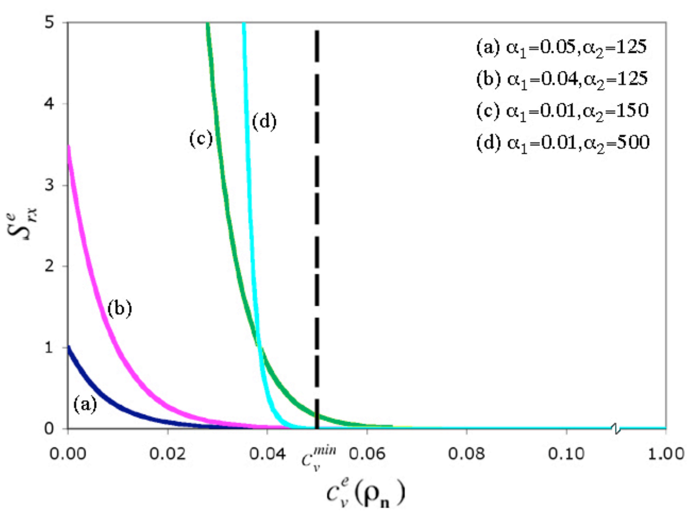

Figure 2. The maximum length scale penalty function grows rapidly if the volume of voids within Ω_{rx}^{e} is less than c_v^{min}. The curve moves from (a) to (b) by decreasing α_1 and grows more rapidly from (c) to (d) by increasing α_2.

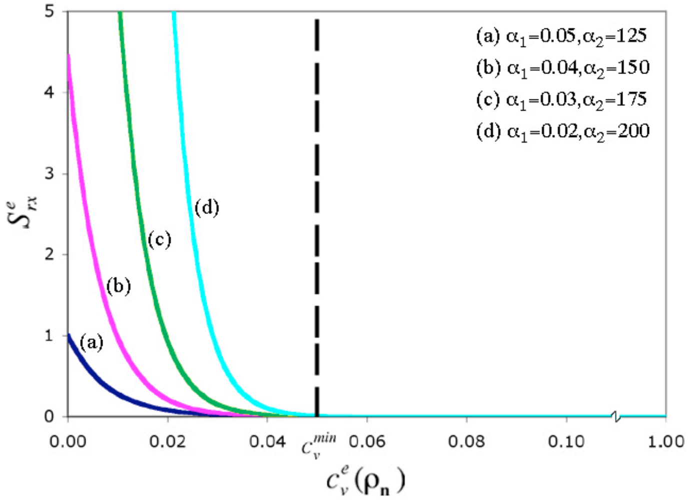

Figure 3. Applying continuation methods to the maximum length scale penalty function: the curve shifts from (a) to (d) by changing parameters α_1 and α_2 as the optimization algorithm progresses.

$$\begin{aligned} &\min_{\boldsymbol{\rho}_\mathbf{n}} \quad -\Pi(\boldsymbol{\rho}_\mathbf{n}) + S_{rx}(\boldsymbol{\rho}_\mathbf{n}) \\ &\text{subject to:} \quad \sum_{e\in\Omega} \rho^e(\boldsymbol{\rho}_\mathbf{n}) v^e \le V \\ &\qquad\qquad \rho_n^{min} \le \rho_n \le 1 \quad \forall\, n \in \Omega \end{aligned} \tag{9}$$

where V is the allowable volume of material and Π is the minimum potential energy expressed as:

$$\Pi(\boldsymbol{\rho}_\mathbf{n}) = \min_{\mathbf{u}} \left(\frac{1}{2}\mathbf{u}^{\mathrm{T}} \mathbf{K}(\boldsymbol{\rho}_\mathbf{n})\, \mathbf{u} - \mathbf{f}^{\mathrm{T}}\mathbf{u} \right) = \min_{\mathbf{u}} \left(\frac{1}{2} \sum_{e\in\Omega} (\rho^e(\boldsymbol{\rho}_\mathbf{n}))^p\ \mathbf{u}^{e\,\mathrm{T}} \mathbf{k}_0^e\, \mathbf{u}^e - \mathbf{f}^{\mathrm{T}}\mathbf{u} \right) \tag{10}$$

where $\mathbf{u}$ are the nodal displacements, $\mathbf{f}$ are the applied nodal loads, and $\mathbf{K}$ is the global stiffness matrix. The expression on the right computes the total strain energy as a summation of element strain energies with $\mathbf{u}^e$ as the nodal displacements corresponding to element e, $\mathbf{k}_0^e$ the element stiffness matrix of a solid element, and the exponent p penalizing intermediate volume fractions as per the SIMP method[6].

Eq. (9) is solved using a nested approach as in Reference 4. Displacements are found by solving the minimum potential energy problem Eq. (10) for a given set of volume fractions ρ^e (and ρ_n). These displacements are then held constant and the optimization problem Eq. (9) is solved to find the optimal nodal volume fractions. This iterative process continues until he algorithm has converged. Eq. (9) is solved using the Method of Moving Asymptotes (MMA)[8] to create convex sub-problems. These sub-problems are solved using an interior point algorithm[9].

III. Maximum Length Scale Results

To demonstrate the capability of the maximum length scale scheme, problems previously solved Reference 4, where only minimum length scale was enforced, are now solved with both minimum and maximum length scales prescribed. In particular, the cantilever beam problem shown in Figure 4 and MBB-beam problem (with deflection and stress constraints relaxed) shown in Figure 5. The results presented below were created using c_v^{min} = 2% and the parameters used in continuation methods were initialized as follows[4]: $\beta = 1$, $p = 1$ (Ref. 4), $\eta = p$, $\alpha_1 = 2\,c_v^{min}$, and $\alpha_2 = 150$. The lower bound for α_1 was set to 0.005. Material properties were Young's modulus E = 1 and Poisson's ratio $\nu = 0.25$. The allowable volume of material V is 50% of the design domain volume, initially distributed uniformly, and the problem was solved using 4-node quadrilateral elements.

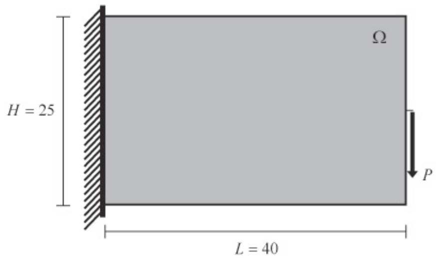

Figure 4. The cantilever beam problem.

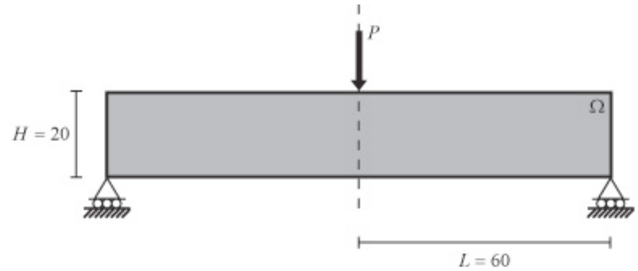

Figure 5. The MBB-beam problem.

Figure 6 demonstrates how the cantilever beam solution for d_{min} = 2 units changes when a relevant maximum length scale is prescribed. The topology shown on the left is the original solution with circles placed at locations where the length scale of structural members is six units (or greater). The solution on the right was created using the maximum length scale penalty function with d_{max} = 6 units. This requirement is clearly satisfied as the design incorporates voids in regions where the maximum length scale would have been violated. Most notably, the center section has split into two smaller sections that are connected to the exterior of the beam by thinner members, each carrying less load than the original larger member design. Note that the portions of the beams closest to the load are nearly identical as maximum length scale was not violated at these locations in the original design.

Figures 7a and 7b contain the original solution to the MBB problem (d_{min} = 2) and the solution with maximum length scale constraint (d_{min} = 2, d_{max} = 6), respectively. The circle in Figure 7a highlights where maximum length scale would be violated. This area and consequently neighboring areas have been redesigned as shown in Figure 7b. More members of smaller length scale extend to the interior of the beam in order to optimally satisfy the maximum length scale requirement. Interestingly, two smaller members in the upper right portion of the original topology have joined to form a single member in the modified design. This is an unexpected result, perhaps suggesting that one of these designs is a local (not global) minimum. Also of note is the region in the lower left corner Figure 7b where minimum length scale appears to be violated. This is caused by the aggressive pursuit of 0-1 solutions. When the nonlinear projection function approaches the Heaviside function (*e.g.*, β > 1000), a single node with large

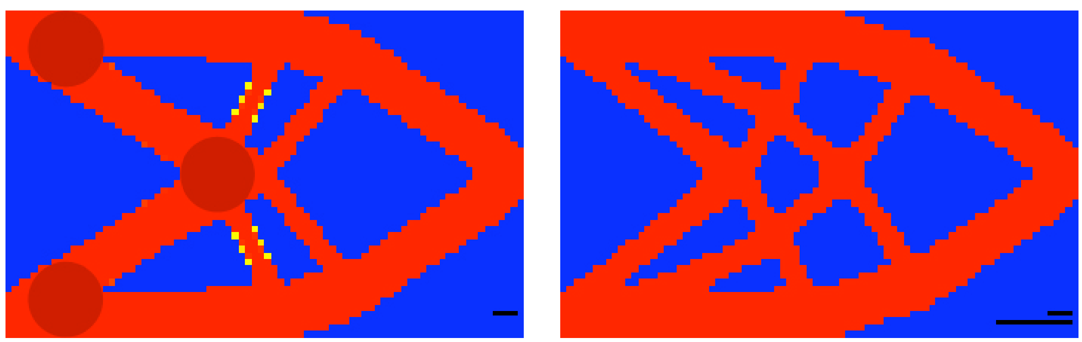

Figure 6. The cantilever beam problem for d_{min} = 2 without maximum length scale control (left) and with maximum length scale of d_{max} = 6 (denoted by the longer black bar) (right). The circles in design on left are of diameter 6, showing where length scale would be violated.

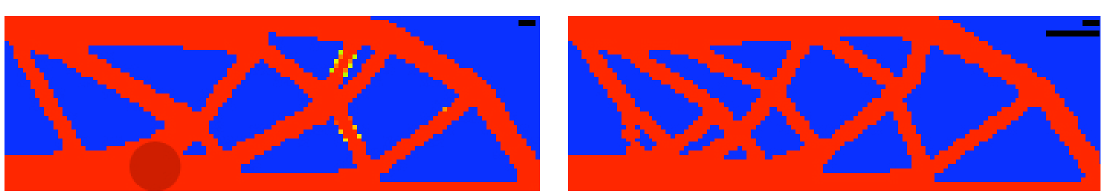

Figure 7. The MBB-beam problem for d_{min} = 2 without maximum length scale control (left) and with maximum length scale of d_{max} = 6 (denoted by the longer black bar) (right). The circles in design on left are of diameter 6, showing where length scale would be violated.

volume fraction is sufficient to form a circular region of material with diameter d_{min}. In this case, just the edges of the circular region touch neighboring structural members. This situation can be avoided by decreasing the maximum allowable value of β.

The cantilever beam and MBB-beam problems were also solved for a minimum length scale of 1 unit and varying magnitudes of maximum length scale. The solutions contained in Figures 8 and 9 further demonstrate that the penalty function ensures maximum length scale compliant topologies that appear to be quite interesting, intricate designs. It is noted, however, that the nonconvexity of the maximum length scale penalty function increases the difficulty of finding the global optimum. Therefore, it must be acknowledged that the solutions presented may be local minima.

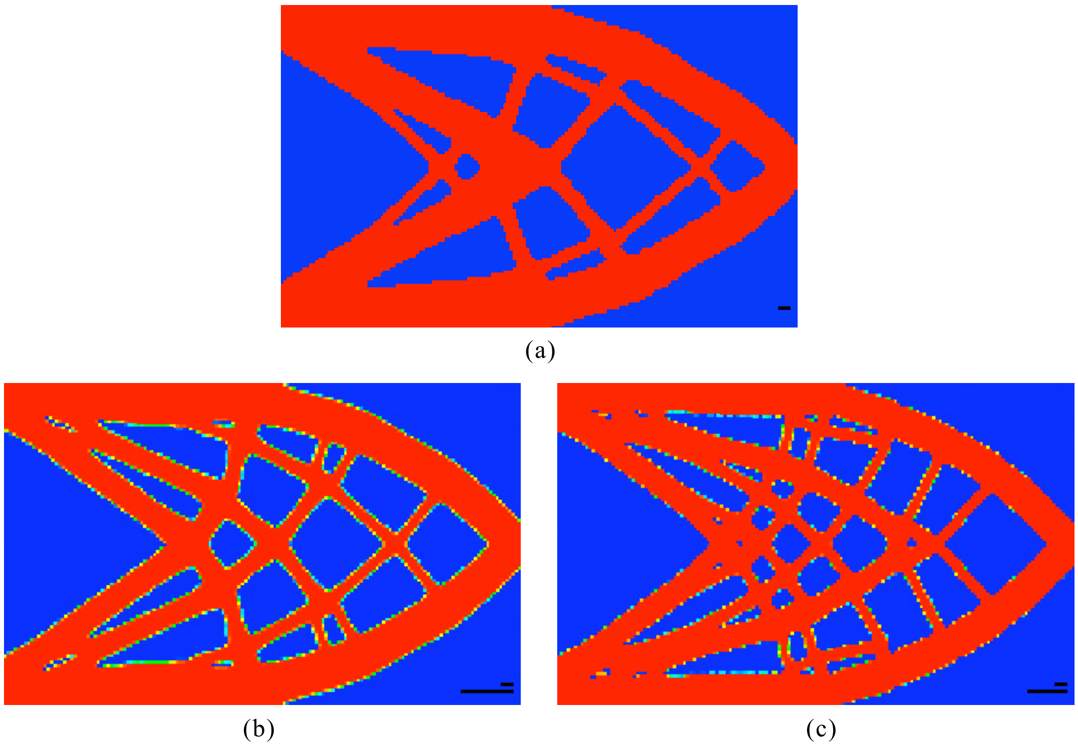

Figure 8. The cantilever beam problem for d_{min} = 1 (a) without maximum length scale control, (b) with maximum length scale of d_{max} = 4, and (c) with maximum length scale of d_{max} = 3. The short and long black bars represent d_{min} and d_{max}, respectively.

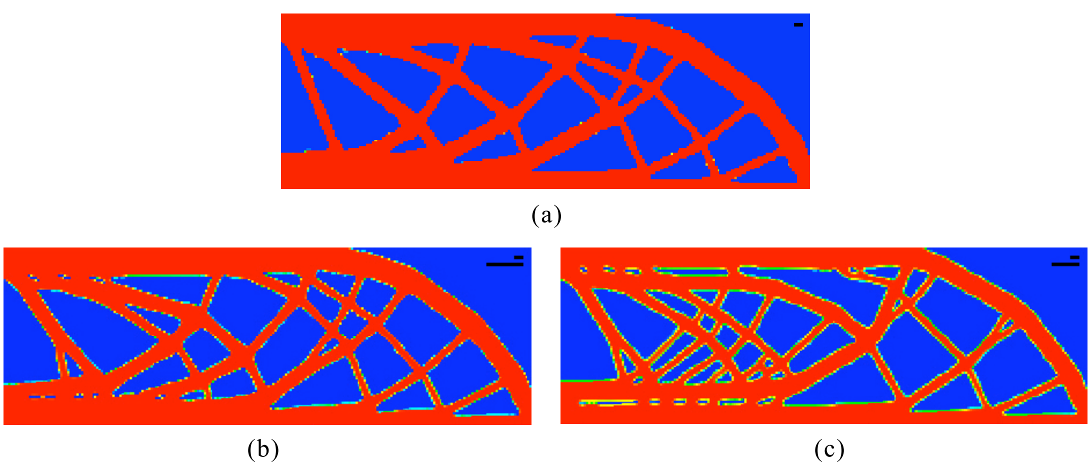

Figure 9. The MBB-beam problem for d_{min} = 1 (a) without maximum length scale control, (b) with maximum length scale of d_{max} = 4, and (c) with maximum length scale of d_{max} = 3. The short and long black bars represent d_{min} and d_{max}, respectively.

IV. Concluding Remarks

This paper proposes a technique for achieving maximum length scale in topology optimization. When implemented in conjunction with minimum length scale criterion, the designer gains complete control over member sizes, thereby providing a means for manufacturing constraints and cost to be considered. It also potentially offers the designer some influence over performance-based attributes; for example, structural redundancy in elastic problems. As demonstrated by the results presented here, load carried by a single large member was often redistributed among several smaller members when maximum length scale was enforced. While more work is required before true structural redundancy is achieved, these preliminary results are promising.

Finally, it should be noted that the maximum length scale penalty function is formulated in element volume fractions only and therefore can be applied to any topology optimization problem, independent of the physics governing the problem. For example, the technique could be combined with recent advances in fluid topology optimization[10,11] in order to design maximum efficiency filters with specified pore sizes.

Acknowledgments

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